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Separable Hamiltonian equations on Riemann manifolds and related integrable hydrodynamic systems

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Abstract

A systematic construction of Stäckel systems in separated coordinates and its relation to bi-Hamiltonian formalism are considered. A general form of related hydrodynamic systems, integrable by the Hamilton–Jacobi method, is derived. One-Casimir bi-Hamiltonian case is studied in details and in this case, a systematic construction of related hydrodynamic systems in arbitrary coordinates is presented, using the cofactor method and soliton symmetry constraints.

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1. Introduction

There is a quite well-developed theory of the passage from an integrable, infinite dimensional Hamiltonian system (soliton system) to its various constrained flows which are themselves completely integrable Hamiltonian systems. Actually, by using the Hamilton–Jacobi method with respect to two evolution parameters x and t, N-gap solutions and N-soliton solutions of a given PDE can be constructed directly from solutions of related ODEs (constrained flows) [1–5].

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In the present paper we are interested in, instead of soliton systems, the first order quasi-linear PDEs of the form

$$q_{it} = \sum_{j=1}^{n} w_{ij}(q) q_{jx}, \quad q_i = q_i(x,t), \quad i = 1, \dots, n, \tag{1.1}$$

called *hydrodynamic* or *dispersionless* systems. More precisely, we consider these systems among (1.1), whose general solutions can be obtained from the solutions of the related integrable finite dimensional Hamiltonian systems, like in the case of soliton systems. Such an approach was presented for the first time by Ferapontov and Fordy [7], where authors gave general solutions of appropriate hydrodynamic systems from general solutions of related separable finite dimensional systems. The idea is the following. Consider, for example a completely integrable Hamiltonian system of two degrees of freedom, given by a Hamiltonian function

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{1.2}$$

and an additional constant of motion

$$F = q_2 p_1 p_2 - q_1 p_2^2 + W(q_1, q_2), \tag{1.3}$$

which commutes with H with respect to the canonical Poisson bracket. Note that both functions are quadratic in momenta, which belongs to the class to be considered in this paper. Let x be an evolution parameter of the flow generated by H and t be an evolution parameter of the flow generated by F. The commutativity of the two flows means that we can consider a two-dimensional surface in phase space, parameterized by x and t. On this surface, the equations of motion for q are

$$q_{ix} = \frac{\partial H}{\partial p_i}, \qquad q_{it} = \frac{\partial F}{\partial p_i}, \quad i = 1, 2. \tag{1.4}$$

Eliminating p_1 and p_2 from (1.4), we obtain a system of hydrodynamic type (1.1)

$$q_{1t} = q_2 q_{2x}, \qquad q_{2t} = q_2 q_{1x} - 2 q_1 q_{2x}. \tag{1.5}$$

In this calculation, V and W play no role, so in fact the hydrodynamic system is generated by the geodesic parts of both Hamiltonian functions. The functions V, W for which H and F commute belong to the Stäckel class of parabolic coordinates and are the conserved density and flux for the hydrodynamic system, respectively, as $V_t = W_x$. Moreover, as there is an infinite hierarchy of separable potentials V, W, we have an infinite hierarchy of conserved densities and related fluxes. Hence, solutions of Hamiltonian systems (1.2) with arbitrary separable potential V are simultaneous solutions of hydrodynamic system (1.5).

Remark 1. One can pass to higher order PDEs eliminating one of the q_i but it is necessary to specify a particular form of V. The Hénon–Heiles potential $V = q_1^3 + (1/2) q_1 q_2^2$ leads to the KdV equation for q_1 [7], whilst the quartic potential $V = 16 q_1^4 + 12 q_1^2 q_2^2 + q_2^4$ generates

$$q_{1t} = -\frac{1}{48} \left(\frac{q_{1xx}}{q_1} + 64 q_1^2 \right)_x ,$$

which is not integrable PDE (all the integrable cases of such equations are listed in canonical form in [8]).

Here following this line we consider the problem more systematically. First, we observe that all examples from Ferapontov and Fordy [7] belong to subclass of Stäckel systems considered by Benenti [17], which, as was shown by Ibort et al. [15], are one-Casimir bi-Hamiltonian systems. Hence, applying one-Casimir and developing multi-Casimir bi-Hamiltonian formalism to quadratic in momenta separable systems, we were able to construct in very systematic way related hydrodynamic systems together with their general solutions. Actually, in Section 2 we derive explicitly, in separated coordinates, the general form of Stäckel systems on Riemann (pseudo-Riemann) manifold and the related hydrodynamic systems integrable by Hamilton–Jacobi method. Then, in Section 3, we connect the considered Stäckel systems with a bi-Hamiltonian formalism generalizing one-Casimir case onto multi-Casimir one. In Section 4 we present a one-Casimir bi-Hamiltonian systems in arbitrary coordinates (not necessary canonical), which we call the cofactor Stäckel systems, and related cofactor hydrodynamic counterparts. Then, we present a recipe for the construction of some class of cofactor hydrodynamic systems in the Cartesian coordinate frame. Finally, in Section 5, we use constrained flows of soliton systems for a systematic derivation of hydrodynamic systems from the class considered.

2. From Stäckel Hamiltonians to complete integral of related hydrodynamic systems

All examples discussed in this paper belong to the class of separable systems associated with Stäckel matrices. Actually, in 1893 Stäckel gave the first characterization of the Riemann (pseudo-Riemann) manifold (Q, g) on which the equations of geodesic motion can be solved by separation of variables. He proved that if in a system of orthogonal coordinates (λ, μ) there exists a nonsingular matrix $\varphi = (\varphi_k^l(\lambda_k))$, called a *Stäckel matrix* such that the geodesic Hamiltonians E_r are of the form

$$E_r = \sum_{i=1}^{n} (\varphi^{-1})_r^i \mu_i^2, \tag{2.1}$$

then E_r are functionally independent, pairwise commute with respect to the canonical Poisson bracket and the Hamilton–Jacobi equation associated to E_1 is separable.

Then, Eisenhart gave a coordinate-free representation for Stäckel geodesic motion introducing special family of *Killing tensors*. He proved [6] that the geodesic Hamiltonians can be transformed into a Stäckel form (2.1) if the contravariant metric tensor $G = g^{-1}$ has $(n-1)$ commuting independent contravariant Killing tensors A_r of a second order such that

$$E_r = \sum_{i,j} A_r^{ij} p_i p_j, \tag{2.2}$$

admitting a common system of closed eigenforms α_i

$$(A_r^* - v_r^i G)\alpha_i = 0, \quad \mathrm{d}\alpha_i = 0, \quad i = 1, \dots, n, \tag{2.3}$$

where v_r^i are eigenvalues of (1, 1) Killing tensor $K_r = A_r g$ ($K_r^* = gA_r^*$). In local coordinates q on Q, we have

$$K_r = \sum_{i,j} (K_r)^i_j \frac{\partial}{\partial q_i} \otimes \mathrm{d}q_j, \qquad K_r^* = \sum_{i,j} (K_r)^j_i \, \mathrm{d}q_i \otimes \frac{\partial}{\partial q_j}. \tag{2.4}$$

From now on, separated canonical coordinates will be denoted by (λ, μ) and natural coordinates, not necessarily canonical, by (q, p). For n degrees of freedom, let us consider n Stäckel Hamiltonian functions in separated coordinates in the following form:

$$H_r = \sum_{i=1}^{n} v_r^i g^{ii} \mu_i^2 + V_r(\lambda) = \mu^{\mathrm{T}} K_r G \mu + V_r(\lambda), \quad r = 1, \ldots, n, \tag{2.5}$$

where $\mu = (\mu_1, \ldots, \mu_n)^{\mathrm{T}}$, and $V_r(\lambda)$ are appropriate potentials separable in (λ, μ) coordinates. So, we have an n-dimensional surface parameterized by n Hamiltonian "times" $t_1 = x, t_2, \ldots, t_n$. In this case we have $(1/2)n(n-1)$ hydrodynamic systems (1.1) written down in the form

$$\lambda_{it_k} = w^i_{kl} \lambda_{it_l}, \qquad w^i_{kl} = \frac{v_k^i}{v_l^i}, \quad k > l = 1, 2, \ldots, n-1, \tag{2.6}$$

and the corresponding conservation laws: $V_{lt_k} = V_{kt_l}$. For a given evolution parameter t_k and a "space" coordinate t_l, there are $n-2$ hydrodynamic flows $\lambda_{it_r} = w^i_{kl}\lambda_{it_l}$, $r \neq k \neq l$ which commute with a given flow. It follows directly from the involutivity of (2.5). The requirement that $H = H_1$, and H_k are in involution with respect to the canonical Poisson bracket leads to the equations

$$\partial_i v_k^i = 0 \quad \text{for any} \;\; i = 1, \ldots, n, \tag{2.7}$$

$$\partial_j \ln (g^{ii}) = \frac{\partial_j v_k^i}{v_k^j - v_k^i} \quad \text{for any} \;\; i \neq j, \tag{2.8}$$

$$\partial_i V_k - v_k^i \partial_i V_1 = 0 \quad \text{for any} \;\; i = 1, \ldots, n, \tag{2.9}$$

where $\partial_i := \partial/\partial q_i$. Condition (2.7) means that our system (2.6) is *linearly degenerate*. Cross differentiation of (2.8) gives

$$\partial_l \left(\frac{\partial_j v_k^i}{v_k^j - v_k^i} \right) = \partial_j \left(\frac{\partial_l v_k^i}{v_k^l - v_k^i} \right) \quad \text{for any} \;\; i \neq j \neq l \neq i. \tag{2.10}$$

This last condition is called the "*semi-Hamiltonian*" property [9] in the context of systems of hydrodynamic type. In that sense we consider weakly nonlinear semi-Hamiltonian (WNSH) hydrodynamic systems.

The solution of the system of equations (2.7)–(2.9) is easy to derive if one realizes that the separated coordinates (λ, μ) for a Liouville integrable system have to fulfil the Sklyanin conditions [10]

$$\varphi_i(\lambda_i, \mu_i; H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \tag{2.11}$$

which guarantees the solvability of an appropriate Hamilton–Jacobi equation. For the integrable system (2.5), under the assumption that all functions φ_i are linear with respect to all H_j, conditions (2.11) take the general form

$$f_i(\lambda_i)\mu_i^2 + \gamma_i(\lambda_i) = \sum_{k=1}^{n} \Phi_i^k(\lambda_i)H_k, \quad i = 1, \dots, n, \tag{2.12}$$

known as Stäckel separation conditions, where f_i, γ_i, Φ_i^k are arbitrary smooth functions of its argument and the normalization $\Phi_i^n = 1$, $i = 1, \dots, n$ is assumed. To get the explicit form of $H_k = H_k(\lambda, \mu)$ one has to solve the system of linear equations (2.12). The results are the following:

$$g^{ii} = (-1)^{i+1}\frac{f_i(\lambda_i)\det W^{i1}}{\det W}, \qquad v_r^i = (-1)^{r+1}\frac{\det W^{ir}}{\det W^{i1}}, \tag{2.13}$$

$$V_r = \sum_{i=1}^{n}(-1)^{i+r}\gamma_i(\lambda_i)\frac{\det W^{ir}}{\det W}, \tag{2.14}$$

where

$$W = \begin{pmatrix} \Phi_1^1(\lambda_1) & \Phi_1^2(\lambda_1) & \cdots & \Phi_1^{n-1}(\lambda_1) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi_n^1(\lambda_n) & \Phi_n^2(\lambda_n) & \cdots & \Phi_n^{n-1}(\lambda_n) & 1 \end{pmatrix}, \tag{2.15}$$

and W^{ik} is the $(n-1) \times (n-1)$ matrix obtained from W after we cancel its ith row and kth column. Then the Stäckel matrix φ is given by

$$\varphi = \begin{pmatrix} \dfrac{\Phi_1^1(\lambda_1)}{f_1(\lambda_1)} & \dfrac{\Phi_1^2(\lambda_1)}{f_1(\lambda_1)} & \cdots & \dfrac{\Phi_1^{n-1}(\lambda_1)}{f_1(\lambda_1)} & \dfrac{1}{f_1(\lambda_1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{\Phi_n^1(\lambda_n)}{f_n(\lambda_n)} & \dfrac{\Phi_n^2(\lambda_n)}{f_n(\lambda_n)} & \cdots & \dfrac{\Phi_n^{n-1}(\lambda_n)}{f_n(\lambda_n)} & \dfrac{1}{f_n(\lambda_n)} \end{pmatrix}. \tag{2.16}$$

Notice that for $r = 2$, we reconstructed the result of Ferapontov [11] for the functions v_2^i.

Remark 2. One can observe that for all known separable systems, we have $f_i = f$, $\gamma_i = \gamma$, $\Phi_i^k = \Phi^k$, $i = 1, \dots, n$, so the conditions (2.12) are represented by the separation (spectral) curve

$$f(\lambda)\mu^2 + \gamma(\lambda) = \sum_{k=1}^{n} \Phi^k(\lambda)H_k.$$

Given a Hamiltonian system in canonical separated coordinates (λ, μ) we can linearize the system through a canonical transformation $(\lambda, \mu) \to (b, a)$ in the form $b_i = \partial S/\partial a_i$,

$\mu_i = \partial S/\partial\lambda_i$, where $S(\lambda, a) = \sum_{i=1}^{n} S_i(\lambda_i, a)$ is an additively separated generating function, satisfying the related Hamilton–Jacobi equations

$$H_r\left(\lambda, \frac{\partial S}{\partial \lambda}\right) = a_r, \quad r = 1, \dots, n. \tag{2.17}$$

For the Hamiltonian functions (2.5), fulfilling the Sklyanin conditions (2.12), $S_i(\lambda_i, a)$ are given by a system of ordinary differential equations

$$f_i(\lambda_i)\left(\frac{\mathrm{d}S_i}{\mathrm{d}\lambda_i}\right)^2 + \gamma_i(\lambda_i) = \sum_{k=1}^{n} \Phi_i^k(\lambda_i) a_k, \quad i = 1, \dots, n. \tag{2.18}$$

Then, in (b, a) coordinates the flows are trivial

$$(a_j)_{t_r} = 0, \qquad (b_j)_{t_r} = \delta_{jr}, \tag{2.19}$$

and the implicit form of the trajectories $\lambda_i(t_1, \dots, t_n)$ is

$$\begin{aligned} b_j(\lambda, a) &= \int^{\lambda_1} \frac{\Phi_1^{n-j}(\xi)}{\varphi_1(\xi)}\,\mathrm{d}\xi + \cdots + \int^{\lambda_n} \frac{\Phi_n^{n-j}(\xi)}{\varphi_n(\xi)}\,\mathrm{d}\xi \\ &= t_j + \mathrm{const.}_j, \quad j = 1, \dots, n, \end{aligned} \tag{2.20}$$

where

$$\varphi_i(\xi) = \left(f_i(\xi)\left[\sum_{k=1}^{n} \Phi_i^k(\xi) a_k - \gamma_i(\xi)\right]\right)^{1/2}. \tag{2.21}$$

Multi-component functions $\lambda_i(t_1, \dots, t_n)$ are simultaneous solutions of all dynamics defined by n Hamiltonians (2.5), as well as general solutions of all hydrodynamic systems (2.6) [11].

Of course, we would like to distinguish our separable Stäckel system (2.5) and the related hydrodynamic systems written down in any natural coordinates (q, p). As we know, for one and a half century this problem has been unsolved in general. Quite recently, there appeared two strong formalisms which allow us to construct systematically a transformation to separated coordinates. One formalism is based on Lax representation of the system considered [10] and the other one on a bi-Hamiltonian formalism [12–15]. In what follows we will use some results of the second formalism.

3. Bi-Hamiltonian Stäckel systems in separable coordinates

An important fact is that the Stäckel systems (2.5) are bi-Hamiltonian on the phase space $M = T^*Q \times R^k$, i.e. the cotangent bundle spanned by k additional Casimir coordinates, where $1 \le k \le n$. Let us assume that M is equipped with a linear Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$ of rank $2n$, i.e. a pair of Poisson operators (tensors) $\Pi_i : T^*M \to TM$ each of rank $2n$ such that their linear combination $\Pi_1 - \lambda\Pi_0$ is again a Poisson operator for any $\lambda \in R$ (the operators Π_0 and Π_1 are then said to be compatible). Moreover, let us assume that k Casimirs $H_\lambda^{(i)}$, $i = 1, \ldots, k$ of the pencil Π_λ are polynomials in λ of orders $n_1, \ldots, n_k$

$$H_\lambda^{(i)} = H_0^{(i)} \lambda^{n_i} + H_1^{(i)} \lambda^{n_i - 1} + \cdots + H_{n_i}^{(i)}, \quad i = 1, \ldots, k, \tag{3.1}$$

where $n_1 + \cdots + n_k = n$. By expanding equations $\Pi_\lambda(\mathrm{d}H_\lambda^{(i)}) = 0$, $i = 1, \ldots, k$ in powers of λ and comparing the coefficients of equal powers, we obtain the following bi-Hamiltonian chains

$$\begin{aligned}
&\Pi_0(\mathrm{d}H_0^{(i)}) = 0 \\
&\Pi_0(\mathrm{d}H_1^{(i)}) = \Pi_1(\mathrm{d}H_0^{(i)}) \\
&\vdots \\
&\Pi_0(\mathrm{d}H_{n_i}^{(i)}) = \Pi_1(\mathrm{d}H_{n_i-1}^{(i)}) \\
&0 = \Pi_1(\mathrm{d}H_{n_i}^{(i)}),
\end{aligned} \tag{3.2}$$

where $i = 1, \ldots, k$. Notice that each chain starts with a Casimir of the first Poisson operator and terminates with a Casimir of the second Poisson operator. As follows from (3.2), the functions $H_j^{(i)}$ are in involution with respect to both Poisson structures. If additionally all $H_j^{(i)}$ are functionally independent, then the chains define a Liouville integrable system. Let us introduce the following Casimir coordinates $c_i = H_0^{(i)}$, $i = 1, \ldots, k$. From the definition, a Darboux–Nijenhuis (DN) separated coordinates (λ, μ, c) are the canonical coordinates in which both Poisson structures take the following form:

$$\Pi_0 = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ -I & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & & \end{pmatrix}, \qquad \Pi_1 = \begin{pmatrix} 0 & \Lambda & \dfrac{\partial H_1^{(1)}}{\partial \mu} & \cdots & \dfrac{\partial H_1^{(k)}}{\partial \mu} \\ -\Lambda & 0 & -\dfrac{\partial H_1^{(1)}}{\partial \lambda} & \cdots & -\dfrac{\partial H_1^{(k)}}{\partial \lambda} \\ * & * & & & \\ \vdots & \vdots & & 0 & \\ * & * & & & \end{pmatrix}, \tag{3.3}$$

where I is an $n \times n$ unit matrix, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and the symbol $*$ denotes the elements that make the matrix skew-symmetric.

Now, n Sklyanin conditions (2.11), with n functions $H_j^{(i)}$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, are

$$f_i(\lambda_i)\mu_i^2 + \bar{\gamma}_i(\lambda_i) = \sum_{j=1}^{k} \Psi_i^j(\lambda_i) H_{\lambda_i}^{(j)}, \quad i = 1, \ldots, n, \tag{3.4}$$

where $f_i, \bar{\gamma}_i, \Psi_i^j$ are arbitrary smooth functions of its argument, $H_{\lambda_i}^{(j)}$ are the Casimir polynomials (3.1) evaluated in $\lambda = \lambda_i$

$$H_{\lambda_i}^{(j)} = c_j \lambda_i^{n_j} + H_1^{(j)} \lambda_i^{n_j - 1} + \cdots + H_{n_i}^{(j)}, \tag{3.5}$$

and the normalization $\Psi_i^k = 1$, $i = 1, \dots, n$ is assumed. One can immediately reconstruct conditions (2.12), as $H_1 = H_1^{(1)}$, $H_2 = H_2^{(1)}, \dots, H_n = H_{n_k}^{(k)}$, $\Phi_i^1(\lambda_i) = \lambda_i^{n_1-1}\Psi_i^1(\lambda_i)$, $\Phi_i^2(\lambda_i) = \lambda_i^{n_1-2}\Psi_i^1(\lambda_i), \dots, \Phi_i^{n-1}(\lambda_i) = \lambda_i$, $\gamma_i(\lambda_i) = \bar{\gamma}_i(\lambda_i) - \sum_{j=1}^k \lambda_i^{n_j}\Psi_i^j(\lambda_i)c_j$ and then, from (2.13) we get $H_j^{(i)} = H_j^{(i)}(\lambda, \mu, c)$.

The simplest case of one-Casimir is determined by the following Stäckel conditions:

$$f_i(\lambda_i)\mu_i^2 + \gamma_i(\lambda_i) = c\lambda_i^n + H_1\lambda_i^{n-1} + \cdots + H_n, \quad i = 1, \dots, n. \tag{3.6}$$

Here we put for simplicity $c_1 = c$, $H_k^{(1)} = H_k$. Notice that related hydrodynamic systems, written in λ coordinates, are completely described by v_r^i functions (2.13) which are determined by the r.h.s. of the Stäckel conditions (2.12). Because in one-Casimir case the r.h.s. of Eq. (3.6) is fixed, there is a unique set of functions v_k^i. Actually, as was found in [12],

$$H_r = -\sum_{i=1}^n \frac{\partial\rho_r}{\partial\lambda_i}\frac{f_i(\lambda_i)\mu_i^2 + \gamma_i(\lambda_i)}{\Delta_i} + c\rho_r(\lambda), \quad r = 1, \dots, n, \tag{3.7}$$

where $\Delta_i = \prod_{j\neq i}(\lambda_i - \lambda_j)$, and $\rho_r(\lambda)$ are coefficients of the characteristic polynomial of Λ

$$\det(\lambda I - \Lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \sum_{i=0}^n \rho_i\lambda^i, \tag{3.8}$$

i.e. are Viète polynomials. In the notation of Eq. (2.5) it means that

$$g^{ii} = \frac{f_i(\lambda_i)}{\Delta_i}, \qquad v_r^i = -\frac{\partial\rho_r}{\partial\lambda_i}, \qquad V_r = -\sum_{i=1}^n \frac{\partial\rho_r}{\partial\lambda_i}\frac{\gamma_i(\lambda_i)}{\Delta_i}, \tag{3.9}$$

so $K_r = \mathrm{diag}(v_r^1, \dots, v_r^n)$, $G = \mathrm{diag}(g^{11}, \dots, g^{nn})$ and the related Stäckel matrix φ takes the form

$$\varphi = \begin{pmatrix} \dfrac{\lambda_1^{n-1}}{f_1(\lambda_1)} & \dfrac{\lambda_1^{n-2}}{f_1(\lambda_1)} & \cdots & \dfrac{\lambda_1}{f_1(\lambda_1)} & \dfrac{1}{f_1(\lambda_1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{\lambda_n^{n-1}}{f_n(\lambda_n)} & \dfrac{\lambda_n^{n-2}}{f_n(\lambda_n)} & \cdots & \dfrac{\lambda_n}{f_n(\lambda_n)} & \dfrac{1}{f_n(\lambda_n)} \end{pmatrix}. \tag{3.10}$$

An additional term $c\rho_r(\lambda)$ is related to the new Casimir coordinate and can be absorbed by the potential. Note that the Killing tensors K_r are given by the cofactor representation

$$\mathrm{cof}(\lambda I - \Lambda) = \sum_{i=0}^{n-1} K_{n-i}\lambda^i, \tag{3.11}$$

where $\mathrm{cof}(A)$ stands for the matrix of cofactors, so that $\mathrm{cof}(A)A = (\det A)I$. The cofactor nature of K_r gives immediately the following relation:

$$K_{r+1} = \sum_{k=0}^r \rho_k(\lambda)\Lambda^{r-k}, \tag{3.12}$$

and vice versa, K_r given by the relation (3.12) are of cofactor form (3.11). The functions H_r (3.7) form a single bi-Hamiltonian chain (3.2), where

$$\Pi_0 = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Pi_1 = \begin{pmatrix} 0 & \Lambda & \dfrac{\partial H_1}{\partial \mu} \\ -\Lambda & 0 & -\dfrac{\partial H_1}{\partial \lambda} \\ * & * & 0 \end{pmatrix}. \tag{3.13}$$

Observe that both Poisson structures can be projected onto T^*Q

$$\Theta_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \Theta_1 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \tag{3.14}$$

and

$$N = \theta_1 \theta_0^{-1} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \tag{3.15}$$

is a $(1, 1)$ tensor on T^*Q with a vanishing Nijenhuis torsion. The operator N is just a lift from Q to T^*Q of a $(1, 1)$ tensor Λ

$$\Lambda = \sum_i \lambda_i \frac{\partial}{\partial \lambda_i} \otimes \mathrm{d}\lambda_i \tag{3.16}$$

on Q with a vanishing Nijenhuis torsion [15]. Moreover, as

$$\Lambda^* \,\mathrm{d}\lambda_i = \lambda_i \,\mathrm{d}\lambda_i, \tag{3.17}$$

then

$$\begin{aligned} K^*_{r+1} \,\mathrm{d}\lambda_i &= \sum_{k=0}^{r} \rho_k(\lambda)(\Lambda^*)^{r-k} \,\mathrm{d}\lambda_i \\ &= \sum_{k=0}^{r} \rho_k(\lambda)\lambda_i^{r-k} \,\mathrm{d}\lambda_i = -\frac{\partial \rho_{r+1}}{\partial \lambda_i} \,\mathrm{d}\lambda_i = v^i_{r+1} \,\mathrm{d}\lambda_i, \end{aligned} \tag{3.18}$$

and multiplying both sides of Eq. (3.18) by G we get

$$(GK^*_r - v^i_r G)\,\mathrm{d}\lambda_i = 0 \Leftrightarrow (A^*_r - v^i_r G)\,\mathrm{d}\lambda_i = 0, \quad i = 1, \dots, n, \tag{3.19}$$

i.e. the tensorial Eisenhart realization (2.3) of the Stäckel results. Moreover, if we define

$$\bar{\Lambda} := \tfrac{1}{2}\mu^{\mathrm{T}} \Lambda G \mu, \tag{3.20}$$

then

$$\{\bar{\Lambda}, E_1\}_{\theta_0} = aE_1, \quad a = \mu^{\mathrm{T}} G \frac{\partial(\mathrm{Tr}\, \Lambda)}{\partial \lambda}. \tag{3.21}$$

Λ of such property is called conformal Killing tensor with the associated potential in the form of Tr Λ. Finally, note that H_1 and H_n are related by

$$-\rho_n \Pi_0(\mathrm{d}H_1) = \Pi_1(\mathrm{d}H_n), \tag{3.22}$$

which is known as the quasi-bi-Hamiltonian representation and is just a result of the projection of one-Casimir Poisson pencil onto a symplectic leaf of Π_0 [13,15].

Thus, within the class of one-Casimir Stäckel systems, i.e. cofactor Stäckel systems, infinitely many systems from that class are related to $(1/2)n(n-1)$ hydrodynamic systems (2.6) governed by n Killing matrices K_r (3.11) from geodesic Hamiltonians. For example: $v_1^i = 1$, $v_2^i = \lambda_i - \sum_{k=1}^n \lambda_k$ and $v_n^i = (-1)^n(\prod_{k=1}^n \lambda_k)/\lambda_i$.

There exists a sequence of generic separable potentials $V_r^{(k)}$, $k = \pm 1, \pm 2, \dots$, which can be added to geodesic Hamiltonians, given by the following recursion relation [16]:

$$V_r^{(k+1)} = -V_{r+1}^{(k)} + V_r^{(1)} V_1^{(k)}, \quad V_r^{(1)} = \rho_r, \quad k = 1, 2, \dots, \tag{3.23}$$

and its inverse

$$V_r^{(-k-1)} = -V_{r-1}^{(-k)} + V_r^{(-1)} V_n^{(-k)}, \quad V_r^{(-1)} = \frac{\rho_{r-1}}{\rho_n}, \quad k = 1, 2, \dots . \tag{3.24}$$

Using the notation $V_\lambda = \sum_{j=0}^{n-1} V_{n-j}\lambda^j$, the recursion formulas (3.23) and (3.24) can be written in a compact form

$$V_\lambda^{(k+1)} = \det(\lambda I - \Lambda) V_1^{(k)} - \lambda V_\lambda^{(k)}, \tag{3.25}$$

and

$$V_\lambda^{(-k-1)} = \frac{1}{\lambda}\left(\frac{\det(\lambda I - \Lambda)}{\det \Lambda} V_n^{(-k)} - V_\lambda^{(-k)}\right). \tag{3.26}$$

Potentials $V^{(k)}$ (3.23) and $V^{(-k)}$ (3.24) are generated by the corresponding monomials $\gamma_i(\lambda_i) = \lambda_i^{n+k-1}$ and $\gamma_i(\lambda_i) = \lambda_i^{-k}$ from (3.6). The infinite hierarchy of conservation laws takes the form $(V_r^{(k)})_{t_s} = (V_s^{(k)})_{t_r}$, $k = \pm 1, \pm 2, \dots$.

Now, the complete integral (2.20) of hydrodynamic systems (2.6) with $v_r^i = -(\partial\rho_r/\partial\lambda_i)$ is given by

$$\int^{\lambda_1} \frac{\xi^{n-j}}{\varphi_1(\xi)}\,\mathrm{d}\xi + \cdots + \int^{\lambda_n} \frac{\xi_n^{n-j}}{\varphi_n(\xi)}\,\mathrm{d}\xi = t_j + \mathrm{const.}_j, \quad j = 1, \dots, n, \tag{3.27}$$

where the functions $\varphi_i(\xi)$ are arbitrary. Additionally, one can always construct a $(1 + (n - 1))$-dimensional hydrodynamic system with the solution of the form (3.27) involving all independent variables t_j, $j = 1, \dots, n$ simultaneously.

Example 1. Three-field system $(\lambda_1, \lambda_2, \lambda_3)$.

In this case, we have

$$\rho_1 = -\lambda_1 - \lambda_2 - \lambda_3, \qquad \rho_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \qquad \rho_3 = -\lambda_1\lambda_2\lambda_3, \tag{3.28}$$

hence the following three WNSH hydrodynamic systems in Riemann invariant form are admissible:

$$\lambda_{1t_2} = -(\lambda_2+\lambda_3)\lambda_{1t_1}, \qquad \lambda_{2t_2} = -(\lambda_1+\lambda_3)\lambda_{2t_1}, \qquad \lambda_{3t_2} = -(\lambda_1+\lambda_2)\lambda_{3t_1}, \tag{3.29}$$

$$\lambda_{1t_3} = \lambda_2\lambda_3\lambda_{1t_1}, \qquad \lambda_{2t_3} = \lambda_1\lambda_3\lambda_{2t_1}, \qquad \lambda_{3t_3} = \lambda_1\lambda_2\lambda_{3t_1}, \tag{3.30}$$

$$\lambda_{1t_3} = -\frac{\lambda_2\lambda_3}{\lambda_2+\lambda_3}\lambda_{1t_2}, \qquad \lambda_{2t_3} = -\frac{\lambda_1\lambda_3}{\lambda_1+\lambda_3}\lambda_{2t_2}, \qquad \lambda_{3t_3} = -\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\lambda_{3t_2}. \tag{3.31}$$

The complete integral for all these systems is given by

$$\int^{\lambda_1} \frac{\xi^{3-j}}{\varphi_1(\xi)}\,\mathrm{d}\xi + \int^{\lambda_2} \frac{\xi^{3-j}}{\varphi_2(\xi)}\,\mathrm{d}\xi + \int^{\lambda_3} \frac{\xi^{3-j}}{\varphi_3(\xi)}\,\mathrm{d}\xi = t_j + \mathrm{const.}_j, \qquad j = 1, 2, 3. \tag{3.32}$$

In each of the above cases, only a pair of t_i coordinates is involved, so the third one can be put equal to zero. Notice that the first two equations can be coupled into a $(1+2)$-dimensional hydrodynamic system

$$\begin{aligned}
\lambda_{1t_3} &= -\tfrac{1}{2}(\lambda_2^2+\lambda_3^2)\lambda_{1t_1} - \tfrac{1}{2}(\lambda_2+\lambda_3)\lambda_{1t_2},\\
\lambda_{2t_3} &= -\tfrac{1}{2}(\lambda_1^2+\lambda_3^2)\lambda_{2t_1} - \tfrac{1}{2}(\lambda_1+\lambda_3)\lambda_{2t_2},\\
\lambda_{3t_3} &= -\tfrac{1}{2}(\lambda_1^2+\lambda_2^2)\lambda_{3t_1} - \tfrac{1}{2}(\lambda_1+\lambda_2)\lambda_{3t_2}
\end{aligned}$$

for which (3.32) is an integral involving simultaneously a triple of independent coordinates $t_i, i = 1, 2, 3$.

The two-Casimir case is given by the following Stäckel conditions:

$$\begin{aligned}
f_i(\lambda_i)\mu_i^2 + \gamma_i(\lambda_i) = \Psi_i^1(c_1\lambda_i^{n_1} + H_1^{(1)}\lambda_i^{n_1-1} + \cdots + H_{n_1}^{(1)}) + c_2\lambda_i^{n_2}\\
+ H_1^{(2)}\lambda_i^{n_2-1} + \cdots + H_{n_2}^{(2)},
\end{aligned} \tag{3.33}$$

where $i = 1, \ldots, n$ and $n_1 + n_2 = n$. Because v_r^i are determined by the r.h.s. of (3.33), involving arbitrary functions Ψ_i^1, we have infinitely many sets of v_r^i functions and infinitely many different WNSH hydrodynamic systems written in λ coordinates, integrable by the Hamilton–Jacobi method. A particular example, with $n_1 = 1, n_2 = n-1$ and $\Psi_i^1 = \lambda_i^n$, can be found in [13].

As mentioned before, we are interested in constructing hydrodynamic systems and a hierarchy of conservation laws, written down in some natural coordinates, which are integrable by the Hamilton–Jacobi method. Then, we would like to find an appropriate transformation to Riemann invariant form (2.6) represented by separated coordinates. In the next section, we present some results, based on the known theory of the separable one-Casimir bi-Hamiltonian chains [19–22].

4. Separable cofactor systems in arbitrary coordinates

As most relations presented in the previous section, although derived in separated coordinates, are of tensorial form, i.e. coordinate-free form so they are valid in arbitrary coordinate frame spanning Q. First, let us restrict our considerations to a class of one-Casimir bi-Hamiltonian systems written in arbitrary canonical coordinates governed by a nondegenerate point transformation between λ and q coordinates. After the transformation, one gets the following Poisson structures

$$\Pi_0 = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Pi_1 = \begin{pmatrix} 0 & L(q) & \dfrac{\partial H_1}{\partial p} \\ -L^{\mathrm{T}}(q) & F(q,p) & -\dfrac{\partial H_1}{\partial q} \\ * & * & 0 \end{pmatrix}, \tag{4.1}$$

where $F_{ij} = (\partial/\partial q_i)(Lp)_j - (\partial/\partial q_j)(p^{\mathrm{T}}L)_i$ and a Casimir of the pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$ of the following form:

$$\begin{aligned} H_\lambda(q,p,c) &= \sum_{i=0}^{n} H_{n-i}(q,p,c)\lambda^i \\ &= p^{\mathrm{T}}\operatorname{cof}(I\lambda - L)G(q)p + V_\lambda^{(\pm k)}(q) + c\det(\lambda I - L), \end{aligned} \tag{4.2}$$

where separable potentials are generated by the recursions

$$V_\lambda^{(k+1)}(q) = \det(\lambda I - L)V_1^{(k)}(q) - \lambda V_\lambda^{(k)}(q), \tag{4.3}$$

$$V_\lambda^{(-k-1)}(q) = \frac{1}{\lambda}\left(\frac{\det(\lambda I - L)}{\det L}V_n^{(-k)}(q) - V_\lambda^{(-k)}(q)\right). \tag{4.4}$$

The geodesic Hamiltonians are as follows:

$$E_r(q,p) = p^{\mathrm{T}}A_r(q)p = p^{\mathrm{T}}K_r(q)G(q)p, \qquad \sum_{i=0}^{n-1} K_{n-i}(q)\lambda^i = \operatorname{cof}(\lambda I - L(q)) \tag{4.5}$$

with a contravariant metric tensor $G(q)$ and

$$K_r = \sum_{k=0}^{r} \rho_k(q)L^{r-k}, \tag{4.6}$$

where $\rho_r(q)$ are coefficients of a characteristic polynomial of $L(q)$. Note that cofactor Stäckel systems are exactly those considered by Benenti et al. [15,17,18].

Then, the hydrodynamic equations in q representation take the form

$$q_{t_j} = K_j K_i^{-1} q_{t_i}, \qquad j > i = 1, \ldots, n-1. \tag{4.7}$$

This class of hydrodynamic systems will be called *cofactor hydrodynamic systems*. Observe that to get cofactor hydrodynamic systems (4.7), we only need a (1, 1) conformal Killing tensor $L(q)$. Moreover, the separated coordinates λ_i, $i = 1, \dots, n$, in which the system (4.7) takes the WNSH form (2.6), are given by the roots of

$$\det(\lambda I - L) = 0. \tag{4.8}$$

Unfortunately, there is no systematic method of constructing L tensors in a general case. Nevertheless, recently some progress has been made in a special case of flat spaces and Cartesian q coordinates.

Let us consider a class of one-Casimir bi-Hamiltonian systems on a flat space $Q = R^n$, introduced recently in [19–21]. Here we briefly review the results which are important for our construction. Let $q = (q_1, \dots, q_n)^{\mathrm{T}}$ be a set of Cartesian coordinates and A, an $n \times n$ matrix, whose elements fulfil the following equations:

$$\partial_i A_{jk} + \partial_g A_{ki} + \partial_k A_{ij} = 0, \quad i, j, k = 1, \dots, n. \tag{4.9}$$

Eq. (4.9) imply that the matrix A is a Killing tensor. An important class of solutions of these equations have the form [21]

$$A = \mathrm{cof}(G) \tag{4.10}$$

with

$$G = \alpha q q^{\mathrm{T}} + \beta q^{\mathrm{T}} + q\beta^{\mathrm{T}} + \gamma, \tag{4.11}$$

where α is a real constant, $\beta = (\beta_1, \dots, \beta_n)^{\mathrm{T}}$ is a column vector of constants and where γ is a symmetric $n \times n$ matrix. One can show that for $n = 2$ it is the general solution of (4.9) [11].

Now, let

$$\tilde{G} = \tilde{\alpha} q q^{\mathrm{T}} + \tilde{\beta} q^{\mathrm{T}} + q\tilde{\beta}^{\mathrm{T}} + \tilde{\gamma} \tag{4.12}$$

be another matrix of the form (4.11) (we assume that at least one of the matrices G and $\tilde{G}$ is nonconstant), then, all matrices A_i, $i = 1, \dots, n$, defined as coefficients in the polynomial expansion of $\mathrm{cof}(\tilde{G} + \lambda G)$ with respect to the real parameter λ

$$\mathrm{cof}(\tilde{G} + \lambda G) = \sum_{i=0}^{n-1} A_{n-i} \lambda^i \tag{4.13}$$

with $A_1 = \mathrm{cof}(G)$, $A_n = \mathrm{cof}(\tilde{G})$, are Killing tensors.

Let us assume that over some region of Q $\det G \neq 0$. Then $L = -\tilde{G}G^{-1}$, considered as $(1, 1)$ tensor field on $Q = R^n$, has a vanishing Nijenhuis torsion, and so it can always be diagonalized. Moreover, on T^*Q we define n functions

$$E_i = p^{\mathrm{T}} A_i p, \quad i = 1, \dots, n, \tag{4.14}$$

and a tensor

$$\Theta = \begin{pmatrix} 0 & -G \\ G & F \end{pmatrix}, \tag{4.15}$$

where $(p_1, \ldots, p_n)^{\mathrm{T}}$ are momenta coordinates and an $n \times n$ matrix F is defined by

$$F = Np^{\mathrm{T}} - pN^{\mathrm{T}}, \quad N = \alpha q + \beta. \tag{4.16}$$

Theorem 3. *Assuming that over some region of Q the operator L has n functionally independent nonzero eigenvalues*:

(i) *Θ is a Poisson tensor of rank $2n$.*
(ii) *E_i are functionally independent and in involution with respect to Θ.*

Proof. Operator Θ is skew-symmetric and the Jacobi identity can be proved by inspection. Moreover $\det G \neq 0$ guarantee its maximal rank. On the other hand, the cofactor form of geodesic Hamiltonians together with functional independence of eigenvalues of L operator means that we have the Eisenhart representation and hence the Stäckel ones in separated coordinates. □

Obviously, we have Liouville integrable systems for geodetic motions, written in non-canonical coordinates (q, p). Choosing $G = -I$ we get a special case of canonical representations. Before we separate the system, let us construct related hydrodynamic systems. Equations of motion for q coordinates are as follows:

$$q_{t_i} = -2GA_i p, \quad i = 1, \ldots, n, \tag{4.17}$$

hence, elimination of p coordinates, together with the relation $A_1 = \mathrm{cof}(G)$, leads to the following cofactor hydrodynamic systems:

$$q_{t_j} = A_1^{-1}A_j A_i^{-1}A_1 q_{t_i} = GA_j A_i^{-1}G^{-1}q_{t_i}, \quad j > i = 1, \ldots, n-1. \tag{4.18}$$

In particular case $t_i = t_1 = x$, we have

$$q_{t_j} = A_1^{-1}A_j q_x = \frac{1}{\det G}GA_j q_x, \quad j = 2, \ldots, n. \tag{4.19}$$

To find separated coordinates for the geodesic Hamiltonians (4.14), we have to put them into a bi-Hamiltonian form. It can be done on the extended phase space $M = T^*Q \times R$ with local coordinates (q, p, c). Let us introduce functions $D_j(q)$ as coefficients in the polynomial expansion of $\det(\tilde{G} + \lambda G)$

$$\sum_{i=0}^{n} D_{n-i}(q)\lambda^i = \det(\tilde{G} + \lambda G) \tag{4.20}$$

so that $D_0 = \det G$ and $D_n = \det \tilde{G}$.

Theorem 4. *Functions*

$$H_r = E_r + c\frac{D_r}{D_0}, \quad r = 0, \ldots, n, \quad E_0 = c, \tag{4.21}$$

constitute a bi-Hamiltonian chain with respect to a pair of compatible Poisson structures

$$\Pi_0 = \begin{pmatrix} 0 & -G & 0 \\ G & -F & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Pi_1 = \begin{pmatrix} 0 & \tilde{G} & -2\det(G)p \\ -\tilde{G} & \tilde{F} & 2(\tilde{N}+LN)c \\ * & * & 0 \end{pmatrix}, \tag{4.22}$$

which starts with the Casimir $H_0 = c$ of the first Poisson structure Π_0 and terminates with the Casimir $H_n = E_n + c(D_n/D_0)$ of the second Poisson structure Π_1. The last column of Π_1 is a first vector field from the hierarchy: $\Pi_0(\mathrm{d}H_1)$.

Having a bi-Hamiltonian chain, one can systematically construct separated coordinates [22]. The result is given as follows.

Theorem 5. *For a geodesic Hamiltonian system (4.14) and (4.15) the separated coordinates $\lambda_i(q)$ are the roots of the equation*

$$\det(\lambda I - L) = 0 \Leftrightarrow \det(\lambda G + \tilde{G}) = 0, \tag{4.23}$$

and related momenta $\mu_i(q, p)$ are given by the equations

$$\mu_i(q, p) = \frac{1}{2}\frac{\Omega^{\mathrm{T}}\operatorname{cof}(\tilde{G} + \lambda_i(q)G)p}{\Omega^{\mathrm{T}}\operatorname{cof}(\tilde{G} + \lambda_i(q)G)\Omega}, \qquad i = 1, \ldots, n, \tag{4.24}$$

where $\Omega = (\tilde{N} + LN)$ and $\tilde{N} = \tilde{\alpha}q + \tilde{\beta}$. The Poisson operators (4.22) attain the form

$$\Pi_0 = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Pi_1 = \begin{pmatrix} 0 & \Lambda & \dfrac{\partial H_1}{\partial \mu} \\ -\Lambda & 0 & -\dfrac{\partial H_1}{\partial \lambda} \\ * & * & 0 \end{pmatrix}, \tag{4.25}$$

while the Hamiltonians (4.21) have the form

$$H_r = -\sum_{k=1}^{n} \frac{\partial \rho_r}{\partial \lambda_k}\frac{f_k(\lambda_k)}{\Delta_k}\mu_k^2 + c\rho_r(\lambda), \qquad r = 1, \ldots, n. \tag{4.26}$$

The above theorem ensures us that under the transformation given by the roots of (4.23), cofactor hydrodynamic systems (4.18) turn into the WNSH form (2.6), where $(K_r)^i_j = \delta_{ij}v^j_r = -(\partial\rho_r/\partial\lambda_i)$. To complete the picture from the point of view of Stäckel systems, let us end this consideration with the following result.

Proposition 6. *The generic separable potentials of the cofactor Stäckel system (4.21) and (4.22) are given by the recursion formulas [21]*

$$V_\lambda^{(k+1)}(q) = \frac{\det(\lambda G + \tilde{G})}{\det G} V_1^{(k)}(q) - \lambda V_\lambda^{(k)}(q) = \det(\lambda I - L)V_1^{(k)}(q) - \lambda V_\lambda^{(k)}(q), \tag{4.27}$$

$$V_\lambda^{(-k-1)}(q) = \frac{1}{\lambda}\left(\frac{\det(\lambda G + \tilde{G})}{\det \tilde{G}} V_n^{(-k)}(q) - V_\lambda^{(-k)}(q)\right)$$
$$= \frac{1}{\lambda}\left(\frac{\det(\lambda I - L)}{\det L} V_n^{(-k)} - V_\lambda^{(-k)}\right), \tag{4.28}$$

where $V_\lambda = \sum_{i=0}^{n-1} V_{n-i}\lambda^i$ *and the potential* V_r *belongs to the appropriate Hamiltonian* E_r (4.21).

In the canonical coordinates $G = -I$, $\tilde{G} = L$ and we reconstruct the potentials (4.3) and (4.4).

Example 2. A two field cofactor hydrodynamic system is defined by

$$G = \begin{pmatrix} 1 & q_1 \\ q_1 & 2q_2 \end{pmatrix}, \qquad \tilde{G} = \begin{pmatrix} q_1^2 + 1 & q_1 q_2 \\ q_1 q_2 & q_2^2 \end{pmatrix}.$$

We have

$$A_1 = \begin{pmatrix} 2q_2 & -q_1 \\ -q_1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} q_2^2 & -q_1 q_2 \\ -q_1 q_2 & q_1^2 + 1 \end{pmatrix},$$

and hence, form (4.19)

$$q_{1t_2} = \frac{q_2(q_2 - q_1^2)}{2q_2 - q_1^2} q_{1t_1} - \frac{q_1(q_2 - q_1^2 - 1)}{2q_2 - q_1^2} q_{1t_1},$$
$$q_{2t_2} = -\frac{q_2(q_1^2 + 2)}{2q_2 - q_1^2} q_{1t_1} + \frac{q_2(q_1^2 + 2)}{2q_2 - q_1^2} q_{1t_1}.$$

The transformation (4.23) to the WNSH form

$$\lambda_{1t_2} = \lambda_2 \lambda_{1t_1}, \qquad \lambda_{2t_2} = \lambda_1 \lambda_{2t_1},$$

is given by

$$q_1 = -2\frac{\sqrt{-\lambda_1\lambda_2(\lambda_1 + 1)(\lambda_2 + 2)}}{\lambda_1 + \lambda_2 + \lambda_1\lambda_2}, \qquad q_2 = -2\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2 + \lambda_1\lambda_2}.$$

Example 3. Three-field cofactor hydrodynamic systems are defined by

$$G = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \tilde{G} = \begin{pmatrix} 2q_3 & q_2 & q_1 \\ q_2 & 0 & -1 \\ q_1 & -1 & 0 \end{pmatrix}.$$

We have

$$A_1=\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix},\qquad A_2=\begin{pmatrix}0&1&q_1\\1&2q_1&-q_2\\q_1&-q_2&-2q_3\end{pmatrix},$$
$$A_3=\begin{pmatrix}-1&-q_1&-q_2\\-q_1&-q_1^2&2q_3+q_1q_2\\-q_2&2q_3+q_1q_2&-q_2^2\end{pmatrix},$$

and hence, the following hydrodynamic systems:

$$\begin{aligned}
&q_{1t_2}=-q_1q_{1t_1}+q_2q_{2t_1}+2q_3q_{3t_1},\qquad q_{1t_2}=-q_{1t_1}-2q_1q_{2t_1}+q_2q_{3t_1},\\
&q_{1t_2}=-q_2q_{2t_1}-q_1q_{3t_1},\qquad q_{1t_3}=q_2q_{1t_1}-(q_1q_2+2q_3)q_{2t_1}+q_2^2q_{3t_1},\\
&q_{1t_3}=q_1q_{1t_1}+q_1^2q_{2t_1}-(q_1q_2+2q_3)q_{3t_1},\qquad q_{1t_3}=q_{1t_1}-q_1q_{2t_1}+q_2q_{3t_1},\\
&q_{1t_3}=\frac{1}{2}\frac{2q_2^2+3q_1^2q_2+2q_1q_3}{q_3-q_1q_2-q_1^3}q_{1t_2}-\frac{1}{2}\frac{2q_2q_3+2q_1^2q_3+q_1^3q_2}{q_3-q_1q_2-q_1^3}q_{2t_2}\\
&\qquad+\frac{1}{2}\frac{4q_1q_2q_3+2q_2^3+q_3^2+q_1^2q_2^2}{q_3-q_1q_2-q_1^3}q_{3t_2},\ \ldots.
\end{aligned}$$

The transformation (4.23) to the WNSH form (3.29)–(3.31) is given by

$$q_1=-\tfrac{1}{2}\rho_1,\qquad q_2=\tfrac{1}{2}\rho_2-\tfrac{1}{8}\rho_1^2,\qquad q_3=-\tfrac{1}{2}\rho_3+\tfrac{1}{4}\rho_1\rho_2-\tfrac{1}{16}\rho_1^3,$$

where ρ_i are defined by (3.26).

5. Hydrodynamic equations related to constrained flows of soliton systems

As it was mentioned in Section 1, symmetry constraints of soliton systems give us a systematic method of constructing Liouville integrable finite dimensional Hamiltonian systems [23]. Moreover, most of the examples constructed so far, belongs to separable systems, separated either through a bi-Hamiltonian formalism [3,12] or through a spectral curve method [1,2]. If additionally involutive constants of motion are quadratic forms of momenta, then we can again systematically construct related hydrodynamic systems which have a complete integral in the form (3.27).

We illustrate the approach by two examples. The first one is related to the constrained Schrödinger spectral problem of the KdV hierarchy, where the natural coordinates are its n eigenfunctions. Let q_k be an eigenfunction of the KdV Lax operator ∂_x^2+u with an eigenvalue α_k

$$q_{kxx}+uq_k=\alpha_kq_k,\quad k=1,\ldots,n. \tag{5.1}$$

Under the symmetry constraint

$$u_x = \sum_{i=1}^{n} (q_i^2)_x \Rightarrow u = \sum_{i=1}^{n} q_i^2 + c \tag{5.2}$$

of the KdV equation, where c, plays a role of the additional Casimir coordinate, we obtain the Garnier system, well known in the classical mechanics

$$q_{kxx} + q_k \sum_{i=1}^{n} q_i^2 + cq_k = \alpha_k q_k, \quad k = 1, \ldots, n. \tag{5.3}$$

The bi-Hamiltonian representation of (5.3), in canonical coordinates, was found in [24,25], where the second Poisson structure is

$$\begin{pmatrix} 0 & A - \frac{1}{2}qq^{\mathrm{T}} & p \\ \frac{1}{2}qq^{\mathrm{T}} - A & \frac{1}{2}pq^{\mathrm{T}} - \frac{1}{2}qp^{\mathrm{T}} & [A - cI - (q, q)]q \\ * & * & 0 \end{pmatrix}, \tag{5.4}$$

$p_k = q_{kx}$, $k = 1, \ldots, n$, $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ and $(q, q) = q^{\mathrm{T}}q = \sum_{i=1}^{n} q_i^2$. Observe that this is again a cofactor Stäckel type system where $G = -I$ and $\tilde{G} = L = A - (1/2)qq^{\mathrm{T}}$. Geodesic motion is separable in generalized elliptic coordinates $\lambda_1, \ldots, \lambda_n$, defined by the relation

$$1 + \frac{1}{2} \sum_{k=1}^{n} \frac{q_k^2}{z - \alpha_k} = \frac{\prod_{j=1}^{n}(z - \lambda_j)}{\prod_{j=1}^{n}(z - \alpha_j)}, \tag{5.5}$$

which are just the DN coordinates defined by the roots of $\det(\lambda G + \tilde{G}) = 0$ [12]. The Garnier potential is the first nontrivial one from generic potentials generated by the recursion (4.3), all are separable in elliptic coordinates (5.5). The geodesic Hamiltonians are of the form

$$H_{k+1} = \frac{1}{2}(p, A^k p) + \frac{1}{4} \sum_{j=1}^{k} [(q, A^{j-1}q)(p, A^{k-j}p) - (q, A^{j-1}p)(q, A^{k-j}p)],$$

where $k = 0, \ldots, n - 1$ and they allow us to construct related hydrodynamic systems. For $n = 3$ the three cofactor hydrodynamic systems are

$$q_{t_2} = A_2 q_x, \qquad q_{t_3} = A_3 q_x, \qquad q_{t_3} = A_3 A_2^{-1} q_{t_2}, \tag{5.6}$$

where

$$A_2 = \begin{pmatrix} \frac{1}{2}q_2^2 + \frac{1}{2}q_3^2 - \alpha_2 - \alpha_3 & -\frac{1}{2}q_1q_2 & -\frac{1}{2}q_1q_3 \\ -\frac{1}{2}q_1q_2 & \frac{1}{2}q_1^2 + \frac{1}{2}q_3^2 - \alpha_1 - \alpha_3 & -\frac{1}{2}q_2q_3 \\ -\frac{1}{2}q_1q_3 & -\frac{1}{2}q_2q_3 & \frac{1}{2}q_1^2 + \frac{1}{2}q_2^2 - \alpha_1 - \alpha_2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -\frac{1}{2}\alpha_3q_2^2 - \frac{1}{2}\alpha_2q_3^2 + \alpha_2\alpha_3 & \frac{1}{2}\alpha_3q_1q_2 & \frac{1}{2}\alpha_2q_1q_3 \\ \frac{1}{2}\alpha_3q_1q_2 & -\frac{1}{2}\alpha_3q_1^2 - \frac{1}{2}\alpha_1q_3^2 + \alpha_1\alpha_3 & \frac{1}{2}\alpha_1q_2q_3 \\ \frac{1}{2}\alpha_2q_1q_3 & \frac{1}{2}\alpha_1q_2q_3 & -\frac{1}{2}\alpha_2q_1^2 - \frac{1}{2}\alpha_1q_2^2 + \alpha_1\alpha_2 \end{pmatrix}.$$

The system (5.6) takes a WNSH form (3.29)–(3.31) in the generalized elliptic coordinates after the following coordinate transformation:

$$\rho_1(\lambda) = -\tfrac{1}{2}(q_1^2 + q_2^2 + q_3^3) + \alpha_1 + \alpha_2 + \alpha_3,$$
$$\rho_2(\lambda) = \tfrac{1}{2}[(\alpha_2+\alpha_3)q_1^2 + (\alpha_1+\alpha_3)q_2^2 + (\alpha_1+\alpha_2)q_3^2] - (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3),$$
$$\rho_3(\lambda) = -\tfrac{1}{2}(\alpha_2\alpha_3 q_1^2 + \alpha_1\alpha_3 q_2^2 + \alpha_1\alpha_2 q_3^2) + \alpha_1\alpha_2\alpha_3.$$

Of course just the KdV case offers us infinitely many Liouville integrable bi-Hamiltonian systems, generated from the symmetry constraint of the KdV equation

$$q_{kxx} + uq_k = \alpha_k q_k, \quad k = 1, \ldots, n, \qquad \frac{\delta h_m}{\delta u} = \sum_{k=1}^{n} \frac{\delta \alpha_k}{\delta u} + c = \sum_{k=1}^{n} q_k^2 + c, \tag{5.7}$$

where h_m is the mth conserved functional of the KdV hierarchy. In each case, we get a cofactor type hydrodynamic system.

Just to demonstrate a vast universality of this approach, our second example is related to another soliton hierarchy, represented by the Jaulent–Miodek spectral problem [26] (a special case of Antonowicz and Fordy spectral problem [27])

$$\begin{pmatrix} q_i \\ p_i \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \alpha_i^2 - u_1\alpha_i - u_0 & 0 \end{pmatrix} \begin{pmatrix} q_i \\ p_i \end{pmatrix}, \quad i = 1, \ldots, n, \tag{5.8}$$

and its mth symmetry constraint

$$\frac{\delta h_m}{\delta u} = \frac{1}{2} \begin{pmatrix} (q, Aq) + c \\ (q, q) \end{pmatrix}. \tag{5.9}$$

Here we consider the case of $m = 4$ [3]

$$h_4 = \tfrac{7}{128}u_1^5 + \tfrac{5}{16}u_1^3 u_0 - \tfrac{5}{2}u_{1x}^2 u_1 + \tfrac{3}{8}u_0^2 u_1 - \tfrac{1}{8}u_{1x}u_{0x}.$$

By introducing the Ostrogradsky coordinates

$$q_{n+1} = u_1, \qquad q_{n+2} = u_0, \qquad p_1 = \frac{\delta H_4}{\delta u_{1x}} = -\tfrac{5}{16}u_1 u_{1x} - \tfrac{1}{8}u_{0x},$$
$$p_2 = \frac{\delta H_4}{\delta u_{0x}} = -\frac{1}{8}u_{1x},$$

Eqs. (5.8) and (5.9) for $m = 4$ are transformed into a canonical finite dimensional Hamiltonian system with the Hamiltonian function

$$H_1 = \tfrac{1}{2}(p, p) - \tfrac{1}{2}(q, Aq) + \tfrac{1}{2}q_{n+1}(q, Aq) + \tfrac{1}{2}q_{n+2}(q, q) + cq_{n+1}$$
$$\quad - 8p_{n+1}p_{n+2} + 10q_{n+1}p_{n+2}^2 - \tfrac{5}{16}q_{n+1}^3 q_{n+2} - \tfrac{3}{8}q_{n+1}q_{n+2}^2 - \tfrac{7}{128}q_{n+1}^5.$$

The bi-Hamiltonian (quasi-bi-Hamiltonian) representation was found in [3,28] with the second Poisson structure Π_0 (Θ_0) in the form (4.1) with $L(q)$ matrix of the size $(n+2) \times (n+2)$

$$L=\begin{pmatrix} A & -\frac{1}{4}q & 0_{n\times 1}\\ 0_{1\times n} & q_{n+1} & 1\\ 2q^{\mathrm{T}} & -\frac{1}{2}q_{n+2}-\frac{15}{8}q_{n+1}^2 & -\frac{3}{2}q_{n+1}\end{pmatrix},$$

which gives us a set of $(n+2)$ component cofactor hydrodynamic systems. Let us look at the three-component case of $n=1$, then

$$L=\begin{pmatrix} \alpha & -\frac{1}{4}q_1 & 0\\ 0 & q_2 & 1\\ 2q_1 & -\frac{1}{2}q_3-\frac{15}{8}q_2^2 & -\frac{3}{2}q_2\end{pmatrix},$$

and $\operatorname{cof}(\lambda I-L)=I\lambda^2+A_2\lambda+A_3$, where

$$A_2=\begin{pmatrix} \frac{1}{2}q_2 & -\frac{1}{4}q_1 & 0\\ 0 & \frac{3}{2}q_2-\alpha & 1\\ 2q_2 & -\frac{15}{8}q_2^2-\frac{1}{2}q_3 & -q_2-\alpha\end{pmatrix},$$

$$A_3=\begin{pmatrix} \frac{3}{8}q_2^2+\frac{1}{2}q_3 & -\frac{3}{8}q_1q_2 & -\frac{1}{4}q_1\\ 2q_2 & -\frac{3}{2}\alpha q_2 & -\alpha\\ -2q_2^2 & \frac{15}{8}\alpha q_2^2-\frac{1}{2}q_1q_2+\frac{1}{2}\alpha q_3 & \alpha q_2\end{pmatrix},$$

and the related cofactor hydrodynamic systems are

$$q_{t_2}=A_2q_x,\qquad q_{t_3}=A_3q_x,\qquad q_{t_3}=A_3A_2^{-1}q_{t_2}. \tag{5.10}$$

The system (5.10) takes a WNSH form (3.29)–(3.31) after the following coordinate transformation:

$$q_1=\frac{(5/2)\alpha^3-5\alpha^2\rho_1(\lambda)+(3/2)\alpha\rho_1^2(\lambda)-\alpha\rho_2(\lambda)-\rho_3(\lambda)}{2\alpha-2\rho_1(\lambda)},$$
$$q_2=2\alpha-2\rho_1(\lambda),\qquad q_3=-\alpha^2+2\alpha\rho_1(\lambda)-3\rho_1^1(\lambda)-2\rho_2(\lambda).$$

6. Concluding remarks

In this paper, developing the ideas of Ferapontov and Fordy [7] and Ibort et al. [15], we have established the relation between the bi-Hamiltonian family of Stäckel systems and the class of hydrodynamic systems whose complete integral is constructed from a complete solution of the related Stäckel family. Moreover, we have found the most general admissible Riemann invariant form of such hydrodynamic systems in separated coordinates. In a particular case of one-Casimir Stäckel family (cofactor Stäckel systems), we also presented some systematic methods of construction of related hydrodynamic systems in arbitrary coordinates and the explicit form of the transformation to the Riemann invariant form. The second of the presented methods reveals a new interesting link between soliton systems and the class of hydrodynamic systems considered. Actually, the link is as follows:

$$\text{soliton system} \rightarrow \text{Stäckel system} \rightarrow \text{hydrodynamic system}$$

and the complete solution to the Stäckel system is a simultaneous particular solution of both soliton and hydrodynamic system.

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